

## Iteration

There are some equations that cannot be solved using 'standard' methods to find an exact solution, but an approximate solution can usually be found using a numerical method based on iteration.

## Bisection

The bisection method is based on finding two values between which the solution lies and then using the midpoint of the values for the next approximation.

### Example

Find an approximate negative solution to the equation  $x^3 - 12x - 7 = 0$ . Give your answer to 1 DP.

### Solution

Let  $f(x) = x^3 - 12x - 7$ , so we are trying to solve  $f(x) = 0$ .

$$f(0) = -7 \text{ and } f(-1) = 4 \Rightarrow \text{the root lies between } 0 \text{ and } -1$$

$$f(-0.5) = -1.125 \Rightarrow \text{the root lies between } -0.5 \text{ and } -1$$

$$f(-0.75) = 1.578125 \Rightarrow \text{the root lies between } -0.5 \text{ and } -0.75$$

$$f(-0.625) = 0.255859 \Rightarrow \text{the root lies between } -0.5 \text{ and } -0.625$$

$$f(-0.5625) = -0.42798 \Rightarrow \text{the root lies between } -0.5625 \text{ and } -0.625$$

Therefore the answer is  $-0.6$ .

*Note that we chose to start with  $x = 0$  and  $x = 1$  but we could have started with any pair of values and still have obtained the same solution.*

### Question 1.1

Using the bisection method, find  $i$  to 3DP if  $i$  satisfies the equation:

$$25(1+i)^{-3} + \frac{20(1-(1+i)^{-2})}{i} = 61.5$$

### Newton-Raphson iteration

The Newton-Raphson iterative formula states that if  $x_n$  is an approximate solution to the equation  $f(x)=0$  then a better approximation is  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ .

#### Example

Using the Newton-Raphson formula, find an approximate positive solution to the equation  $x^5 + 4x^2 = 7$ , giving your answer to 2 DP.

#### Solution

Let  $f(x) = x^5 + 4x^2 - 7$ , so that:

$$f'(x) = 5x^4 + 8x$$

$$f(1) = -2$$

$$f(2) = 41$$

Therefore there is a solution between 1 and 2. Try  $x_1 = 1$ .

$$x_2 = 1.15385$$

$$x_3 = 1.13336$$

$$x_4 = 1.13290$$

Therefore the solution is 1.13 (to 2DP).

*Note that we chose to start with  $x=1$  and  $x=2$  but we could have started with any pair of values on either side of the true answer.*

#### Question 1.2

Using the Newton-Raphson formula, find an approximate solution to the equation:

$$5(1+i)^{-4} + \frac{10(1-(1+i)^{-5})}{i} = 41$$

Give your answer to 3 DP.

## Solutions

### Solution 1.1

When  $i = 0.03$  the LHS gives 61.15, and when  $i = 0.02$  the LHS gives 62.39. Using the bisection method, we get:

| Value of $i$ | Value of equation |
|--------------|-------------------|
| 0.025        | 61.76             |
| 0.0275       | 61.45             |
| 0.02625      | 61.61             |
| 0.026875     | 61.53             |

So the value of  $i$  is 0.027 to 3DP.

*We started with  $i = 0.03$  and  $i = 0.04$  but we could have started with any pair of values which gave answers either side of 62.*

### Solution 1.2

Let  $f(i) = 5(1+i)^{-4} + \frac{10(1-(1+i)^{-5})}{i} - 41$ , so that:

$$\begin{aligned} f'(i) &= -20(1+i)^{-5} + \frac{i \times 10 \times 5(1+i)^{-6} - 10(1-(1+i)^{-5})}{i^2} \\ &= -20(1+i)^{-5} + \frac{50i(1+i)^{-6} - 10 + 10(1+i)^{-5}}{i^2} \end{aligned}$$

The Newton-Raphson formula gives the formula for the next approximation to be:

$$i - \frac{5(1+i)^{-4} + \frac{10(1-(1+i)^{-5})}{i} - 41}{-20(1+i)^{-5} + \frac{50i(1+i)^{-6} - 10 + 10(1+i)^{-5}}{i^2}}$$

$f(0.1) = 0.323$  and  $f(0.11) = -0.747$ , so the root lies between 0.1 and 0.11.

Try  $i_1 = 0.1$ .

The formula gives  $i_2 = 0.1029557$ ,  $i_3 = 0.1029739$ , ie the root is 0.103 to 3 DP.

*We started with  $i = 0.1$  but we could have started with any value close to the solution.*